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Cubic Derivations on Banach Algebras

Abasalt Bodaghi

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Abstract Let A be a Banach algebra and X be a Banach A-bimodule. A mapping $D : A \longrightarrow X$ is a cubic derivation if D is a cubic homogeneous mapping, that is D is cubic and $D(\lambda a) = \lambda^3 D(a)$ for any complex number λ and all $a \in A$, and $D(ab) = D(a) \cdot b^3 + a^3 \cdot D(b)$ for all $a, b \in A$. In this paper, we prove the stability of a cubic derivation with direct method. We also employ a fixed point method to establish of the stability and the superstability for cubic derivations.

Keywords Banach algebra · Cubic derivation · Stability · Superstability

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1 Introduction

In 1940, Ulam [19] posed the following question concerning the stability of group homomorphisms: Under what condition does there is an additive mapping near an approximately additive mapping between a group and a metric group? One year later, Hyers [9] answered the problem of Ulam under the assumption that the groups are Banach spaces. This problem for linear mapping on Banach spaces was solved by J. M. Rassias in [15]. A generalized version of the theorem of Hyers for approximately linear mappings was given by Th. M. Rassias [16]. Subsequently, the stability problems of various functional equation have been extensively investigated by a number of authors (for example, [2], [12] and [14]). In particular, one of the functional equations which has been studied frequently is the cubic functional equation:

$$f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x) \tag{1}$$

Abasalt Bodaghi
Department of Mathematics, Garmsar Branch, Islamic Azad University, Garmsar, Iran
Tel.: +98-232-4225010
E-mail: abasalt.bodaghi@gmail.com

The cubic function $f(x) = ax^3$ is a solution of this functional equation. The stability of the functional equation (1) has been considered on different spaces by a number of writers (for instance, [11] and [17]).

In 2003, Cădariu and Radu applied a fixed point method to the investigation of the Jensen functional equation. They presented a short and a simple proof for the Cauchy functional equation and the quadratic functional equation in [4] and [3], respectively. After that, this method has been applied by many authors to establish of miscellaneous functional equations (see [1], [7] and [13]).

In [8], Eshaghi Gordji et al. introduced the concept of a cubic derivation which is a different notion of the current paper. In fact, they did not consider the homogeneous property of such derivations. In that paper, the authors studied the stability of cubic derivations on commutative Banach algebras. The stability and the superstability of cubic double centralizers and cubic multipliers on Banach algebras has been earlier proved in [10].

In this paper, we prove the stability of cubic derivations on Banach algebras. An example of such derivations is indicated as well. Using a fixed point theorem, we also show that a cubic derivation can be superstable.

2 Stability of cubic derivations

Let A be a Banach algebra. A Banach space X which is also a left A-module is said to be a *left Banach A-module* if there is $k > 0$ such that

$$\|a \cdot x\| \leq k\|a\|\|x\|.$$

Similarly, a right Banach A-module and a Banach A-bimodule are defined. Throughout this paper, we assume that A is a Banach algebra, X is a Banach A-bimodule and denote $\overbrace{A \times A \times ... \times A}^{n-times}$ by A^n. For a natural number n_0, we define $\mathbf{T}_{\frac{1}{n_0}} := \{\mathbf{e^{i\theta}} \ ; \ \mathbf{0} \leq \theta \leq \frac{\mathbf{2\pi}}{\mathbf{n_0}}\}$ and denote $\mathbf{T}_{\frac{1}{n_0}}$ by $\mathbf{T}$ when $n_0 = 1$. We also denote the set of all positive integers numbers, real numbers and complex numbers by $\mathbf{N}$, $\mathbf{R}$ and $\mathbf{C}$, respectively. First let us show by a example that the cubic derivations exist on Banach algebras. Indeed, the following example is taken from [8] with the non-trivial module actions while in the mentioned paper the left module action is zero.

Example Let A be a Banach algebra. Consider

$$\mathcal{T} := \begin{bmatrix} 0 & A & A & A \\ 0 & 0 & A & A \\ 0 & 0 & 0 & A \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then $\mathcal{T}$ is a Banach algebra with the sum and product being given by the usual 4×4 matrix operations and with the following norm:

$$\| \begin{bmatrix} 0 & a_1 & a_2 & a_3 \\ 0 & 0 & a_4 & a_5 \\ 0 & 0 & 0 & a_6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \| = \sum_{j=1}^{6} \|a_j\| \qquad (a_j \in A).$$

So

$$\mathcal{T}^* = \begin{bmatrix} 0 & A^* & A^* & A^* \\ 0 & 0 & A^* & A^* \\ 0 & 0 & 0 & A^* \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

is the dual of $\mathcal{T}$ equipped with the following norm:

$$\| \begin{bmatrix} 0 & f_1 & f_2 & f_3 \\ 0 & 0 & f_4 & f_5 \\ 0 & 0 & 0 & f_6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \| = Max\{\|f_j\| : 0 \le j \le 6\} \qquad (f_j \in A^*).$$

Suppose that $\mathcal{A} = \begin{bmatrix} 0 & a_1 & a_2 & a_3 \\ 0 & 0 & a_4 & a_5 \\ 0 & 0 & 0 & a_6 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathcal{X} = \begin{bmatrix} 0 & x_1 & x_2 & x_3 \\ 0 & 0 & x_4 & x_5 \\ 0 & 0 & 0 & x_6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in \mathcal{T}$ and $\mathcal{F} = \begin{bmatrix} 0 & f_1 & f_2 & f_3 \\ 0 & 0 & f_4 & f_5 \\ 0 & 0 & 0 & f_6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in \mathcal{T}^*$ in which $f_j \in A^*$ and $a_j, x_j \in A$ $(0 \le j \le 6)$. Consider the module actions of $\mathcal{T}$ on $\mathcal{T}^*$ as follows:

$$\langle \mathcal{F}\cdot\mathcal{A}, \mathcal{X}\rangle = \sum_{j=1}^{6} f(a_j x_j), \quad \langle \mathcal{A}\cdot\mathcal{F}, \mathcal{X}\rangle = \sum_{j=1}^{6} f(x_j a_j).$$

Then $\mathcal{T}^*$ is a Banach $\mathcal{T}$-bimodule. Let $\mathcal{G}_0 = \begin{bmatrix} 0 & g_1 & g_2 & g_3 \\ 0 & 0 & g_4 & g_5 \\ 0 & 0 & 0 & g_6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in \mathcal{T}^*$. Define $D : \mathcal{T} \longrightarrow \mathcal{T}^*$ via

$$D(\mathcal{A}) = \mathcal{G}_0 \cdot \mathcal{A}^3 - \mathcal{A}^3 \cdot \mathcal{G}_0 \qquad (\mathcal{A} \in \mathcal{T}).$$

Given $\mathcal{A} = \begin{bmatrix} 0 & a_1 & a_2 & a_3 \\ 0 & 0 & a_4 & a_5 \\ 0 & 0 & 0 & a_6 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathcal{B} = \begin{bmatrix} 0 & b_1 & b_2 & b_3 \\ 0 & 0 & b_4 & b_5 \\ 0 & 0 & 0 & b_6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in \mathcal{T}$, we have

$$\begin{aligned}\langle D(2\mathcal{A}+\mathcal{B}),\mathcal{X}\rangle &= \langle \mathcal{G}_0\cdot(2\mathcal{A}+\mathcal{B})^3-(2\mathcal{A}+\mathcal{B})^3\cdot\mathcal{G}_0,\mathcal{X}\rangle\\ &= g_3((2a_1+b_1)(2a_4+b_4)(2a_6+b_6)x_3)\\ &- g_3(x_3(2a_1+b_1)(2a_4+b_4)(2a_6+b_6)). \end{aligned} \tag{2}$$

Similarly,

$$\begin{aligned}\langle D(2\mathcal{A}-\mathcal{B}),\mathcal{X}\rangle &= g_3((2a_1-b_1)(2a_4-b_4)(2a_6-b_6)x_3)\\ &- g_3(x_3(2a_1-b_1)(2a_4-b_4)(2a_6-b_6)). \end{aligned} \tag{3}$$

On the other hand,

$$\begin{aligned}\langle 2D(\mathcal{A}+\mathcal{B}),\mathcal{X}\rangle &= \langle 2\mathcal{G}_0\cdot(\mathcal{A}+\mathcal{B})^3-2(\mathcal{A}+\mathcal{B})^3\cdot\mathcal{G}_0,\mathcal{X}\rangle\\ &= 2g_3((a_1+b_1)(a_4+b_4)(a_6+b_6)x_3)\\ &- 2g_3(x_3(a_1+b_1)(a_4+b_4)(a_6+b_6)), \end{aligned} \tag{4}$$

and

$$\begin{aligned}\langle 2D(\mathcal{A}-\mathcal{B}),\mathcal{X}\rangle &= \langle 2\mathcal{G}_0\cdot(\mathcal{A}-\mathcal{B})^3-2(\mathcal{A}-\mathcal{B})^3\cdot\mathcal{G}_0,\mathcal{X}\rangle\\ &= 2g_3((a_1-b_1)(a_4-b_4)(a_6-b_6)x_3)\\ &- 2g_3(x_3(a_1-b_1)(a_4-b_4)(a_6-b_6)). \end{aligned} \tag{5}$$

Also,

$$\begin{aligned}\langle 12D(\mathcal{A}),\mathcal{X}\rangle &= \langle 12\mathcal{G}_0\cdot\mathcal{A}^3-12\mathcal{A}^3\cdot\mathcal{G}_0,\mathcal{X}\rangle\\ &= 12g_3(a_1a_4a_6x_3)-12g_3(x_3a_1a_4a_6). \end{aligned} \tag{6}$$

If follows from (2)-(6) that

$$D(2\mathcal{A}+\mathcal{B})+D(2\mathcal{A}-\mathcal{B})=2D(\mathcal{A}+\mathcal{B})+2D(\mathcal{A}-\mathcal{B})+12D(\mathcal{A})$$

for all $\mathcal{A},\mathcal{B}\in\mathcal{T}$. This shows that D is a cubic mapping. It is easy to see that $D(\lambda\mathcal{A})=\lambda^3D(\mathcal{A})$ for all $\mathcal{A}\in\mathcal{T}$ and $\lambda\in\mathbf{C}$. Thus D is a cubic homogeneous mapping. Since $\mathcal{T}^4=\{0\}$, we have $D(\mathcal{A}\mathcal{B})=D(\mathcal{A})\cdot\mathcal{B}^3+\mathcal{A}^3\cdot D(\mathcal{B})=0$ for all $\mathcal{A},\mathcal{B}\in\mathcal{T}$. Hence, D is a cubic derivation.

Now, we are going to prove the stability of cubic derivations on Banach algebras.

Theorem 1 *Suppose that $f:A\longrightarrow X$ is a mapping with $f(0)=0$ for which there exists a function $\varphi:A^4\longrightarrow[0,\infty)$ such that*

$$\widetilde{\varphi}(a,b,c,d):=\sum_{k=0}^{\infty}\frac{1}{8^k}\varphi(2^ka,2^kb,2^kc,2^kd)<\infty \tag{7}$$

$$\|f(2\lambda a+\lambda b)+f(2\lambda a-\lambda b)-2\lambda^3f(a+b)-2\lambda^3f(a-b)-12\lambda^3f(a)\|$$

$$\leq \varphi(a, b, 0, 0) \tag{8}$$

$$\|f(cd) - f(c) \cdot d^3 - c^3 \cdot f(d)\| \leq \varphi(0, 0, c, d) \tag{9}$$

for all $\lambda \in \mathbf{T}_{\frac{1}{\mathbf{n_0}}}$ *and all* $a, b, c, d \in A$*. Also, if for each fixed* $a \in A$ *the mappings* $t \mapsto f(ta)$ *from* $\mathbf{R}$ *to* X *is continuous, then there exists a unique cubic derivation* $D : A \longrightarrow X$ *satisfying*

$$\|f(a) - D(a)\| \leq \frac{1}{16}\widetilde{\varphi}(a, 0, 0, 0) \tag{10}$$

for all $a \in A$*.*

Proof Putting $b = 0$ and $\lambda = 1$ in (8) , we have

$$\|\frac{1}{8}f(2a) - f(a)\| \leq \frac{1}{16}\varphi(a, 0, 0, 0) \tag{11}$$

for all $a \in A$ (indeed $1 \in \mathbf{T}_{\frac{1}{\mathbf{n}}}$ for all $n \in \mathbf{N}$). We replace a by $2a$ in (11) and continue this method to get

$$\left\|\frac{f(2^n a)}{8^n} - f(a)\right\| \leq \frac{1}{16}\sum_{k=0}^{n-1}\frac{\varphi(2^k a, 0, 0, 0)}{8^k} \tag{12}$$

On the other hand, we can use induction to obtain

$$\left\|\frac{f(2^n a)}{8^n} - \frac{f(2^m a)}{8^m}\right\| \leq \frac{1}{16}\sum_{k=m}^{n-1}\frac{\varphi(2^k a, 0, 0, 0)}{8^k} \tag{13}$$

for all $a \in A$, and $n > m \geq 0$. It follows from (7) and (13) that sequence $\left\{\frac{f(2^n a)}{8^n}\right\}$ is Cauchy. Since A is complete, this sequence convergence to the map D, that is

$$\lim_{n\to\infty}\frac{f(2^n a)}{8^n} = D(a) \tag{14}$$

Taking the limit as n tend to infinity in (12) and applying (14), we can see that the inequality (10) holds. Now, replacing a, b by $2^n a, 2^n b$, respectively in (8), we get

$$\|\frac{f(2^n(2\lambda a + b))}{8^n} - \frac{f(2^n(2\lambda a - b))}{8^n} - 2\lambda^3\frac{f(2^n(a + b))}{8^n}$$
$$-2\lambda^3\frac{f(2^n(a - b))}{8^n} - 12\lambda^3\frac{f(2^n a)}{8^n}\| \leq \frac{\varphi(2^n a, 2^n b, 0, 0)}{8^n}.$$

Letting the limit as $n \longrightarrow \infty$, we obtain

$$D(2\lambda a + \lambda b) + D(2\lambda a - \lambda b) = 2\lambda^3 D(a + b) + 2\lambda^3 D(a - b) + 12\lambda^3 f(a) \tag{15}$$

for all $a, b \in A$ and all $\lambda \in \mathbf{T}_{\frac{1}{\mathbf{n_0}}}$. If follows from (15) that D is a cubic mapping when $\lambda = 1$. Letting $b = 0$ in (15), we get $D(\lambda a) = \lambda^3 D(a)$ for all $a \in A$ and $\lambda \in \mathbf{T}_{\frac{1}{\mathbf{n_0}}}$. Now, let $\lambda = e^{i\theta} \in \mathbf{T}$. We set $\lambda_0 = e^{\frac{i\theta}{n_0}}$, thus λ_0 belongs to $\mathbf{T}_{\frac{1}{\mathbf{n_0}}}$ and $D(\lambda a) = D(\lambda_0^{n_0} a) = \lambda_0^{3n_0} D(a) = \lambda^3 D(a)$ for all $a \in A$. Under the assumption that $f(ta)$ is continuous in $t \in \mathbf{R}$ for each fixed $a \in A$, by the same reasoning as in the proof of [5], $D(\lambda a) = \lambda^3 D(a)$ for all $\lambda \in \mathbf{R}$ and $a \in A$. So,

$$D(\lambda a) = D(\frac{\lambda}{|\lambda|} \ |\lambda| a) = \frac{\lambda^3}{|\lambda|^3} D(|\lambda| a) = \frac{\lambda^3}{|\lambda|^3} |\lambda|^3 D(a) = \lambda^3 D(a),$$

for all $a \in A$ and $\lambda \in \mathbf{C}$ ($\lambda \neq \mathbf{0}$). Therefore, D is cubic homogeneous mapping. If we replace c, d by $2^n c, 2^n d$ respectively in (9), we have

$$\|\frac{f(2^{2n} cd)}{8^{2n}} - \frac{f(2^n c)}{8^n} \cdot d^3 - c^3 \cdot \frac{f(2^n d)}{8^n}\| \leq \frac{\varphi(0,0,2^n c, 2^n d)}{8^{2n}} \leq \frac{\varphi(0,0,2^n c, 2^n d)}{8^n}.$$

for all $c, d \in A$. Taking the limit as $n \longrightarrow \infty$, we get $D(cd) = D(c) \cdot d^3 + c^3 \cdot D(d)$, for all $c, d \in A$. This shows that D is a cubic derivation.

Now, let $D' : A \longrightarrow X$ be another cubic derivation satisfying (10). Then we have

$$\begin{aligned}
\|D(a) - D'(a)\| &= \frac{1}{8^n} \|D(2^n a) - D'(2^n a)\| \\
&\leq \frac{1}{8^n} (\|D(2^n a) - f(2^n a)\| + \|f(2^n a) - D'(2^n a)\|) \\
&\leq \frac{1}{8^{n+1}} \widetilde{\varphi}(2^n a, 0, 0, 0) \\
&= \frac{1}{8} \sum_{k=0}^{\infty} \frac{1}{8^{n+k}} \varphi(2^{n+k} a, 0, 0, 0) \\
&= \frac{1}{8} \sum_{k=n}^{\infty} \frac{1}{8^k} \varphi(2^k a, 0, 0, 0)
\end{aligned}$$

for all $a \in A$. By letting $n \longrightarrow \infty$ in the preceding inequality, we immediately find the uniqueness of D. This completes the proof.

Corollary 1 *Let δ, r be positive real numbers with $r < 3$, and let $f : A \longrightarrow X$ be a mapping with $f(0) = 0$ such that*

$$\begin{aligned}
&\|f(2\lambda a + \lambda b) + f(2\lambda a - \lambda b) - 2\lambda^3 f(a+b) - 2\lambda^3 f(a-b) - 12\lambda^3 f(a)\| \\
&\qquad\qquad \leq \delta(\|a\|^r + \|b\|^r) \qquad (16)
\end{aligned}$$

$$\|f(cd) - f(c) \cdot d^3 - c^3 \cdot f(d)\| \leq \delta(\|c\|^r + \|d\|^r) \qquad (17)$$

for all $\lambda \in \mathbf{T}_{\frac{1}{\mathbf{n_0}}}$ *and all* $a, b, c, d \in A$. *Then there exists a unique cubic derivation* $D : A \longrightarrow X$ *satisfying*

$$\|f(a) - D(a)\| \leq \frac{\delta}{2(8 - 2^r)}\|a\|^r \tag{18}$$

for all $a \in A$.

Proof It follows from Theorem 1 by taking

$$\varphi(a, b, c, d) = \delta(\|a\|^r + \|b\|^r + \|c\|^r + \|d\|^r).$$

Theorem 2 *Suppose that* $f : A \longrightarrow X$ *is a mapping with* $f(0) = 0$ *for which there exists a function* $\varphi : A^4 \longrightarrow [0, \infty)$ *satisfying (8), (9) and*

$$\widetilde{\varphi}(a, b, c, d) := \sum_{k=1}^{\infty} 8^k \varphi(2^{-k}a, 2^{-k}b, 2^{-k}c, 2^{-k}d) < \infty,$$

for all $a, b, c, d \in A$. *Also, if for each fixed* $a \in A$ *the mappings* $t \mapsto f(ta)$ *from* $\mathbf{R}$ *to* A *is continuous, then there exists a unique cubic derivation* $D : A \longrightarrow X$ *satisfying*

$$\|f(a) - D(a)\| \leq \frac{1}{16}\widetilde{\varphi}(a, 0, 0, 0) \tag{19}$$

for all $a \in A$.

Proof Putting $b = 0$ and $\lambda = 1$ in (8), we have

$$\|f(2a) - 8f(a)\| \leq \frac{1}{2}\varphi(a, 0, 0, 0) \tag{20}$$

for all $a \in A$. We replace a by $\frac{a}{2}$ in (20) to obtain

$$\|f(a) - 8f(\frac{a}{2})\| \leq \frac{1}{2}\varphi(\frac{a}{2}, 0, 0, 0) \tag{21}$$

Using triangular inequality and proceeding this way, we have

$$\left\|f(a) - 8^n f(\frac{a}{2^n})\right\| \leq \frac{1}{16}\sum_{k=1}^{n} 8^k \varphi(\frac{a}{2^k}, 0, 0, 0) \tag{22}$$

If we show that the sequence $\left\{8^n f(\frac{a}{2^n})\right\}$ is Cauchy, then it will be convergent by the completeness of A. For this, replace a by $\frac{a}{2^m}$ in (22) and then multiply both side by 8^m, we get

$$\begin{aligned}\left\|8^m f(\frac{a}{2^m}) - 8^{m+n} f(\frac{a}{2^{m+n}})\right\| &\leq \frac{1}{16}\sum_{k=1}^{n} 8^{k+m} \varphi(\frac{a}{2^{k+m}}, 0, 0, 0) \\ &= \frac{1}{16}\sum_{k=m+1}^{m+n} 8^k \varphi(\frac{a}{2^k}, 0, 0, 0)\end{aligned}$$

for all $a \in A$, and $n > m \geq 0$. Thus the mentioned sequence is convergent to the map D, i.e.,

$$D(a) = \lim_{n\to\infty} 8^n f(\frac{a}{2^n}).$$

Now, similar to the proof of Theorem 1, we can continue the rest of the proof.

Corollary 2 *Let δ, r be positive real numbers with $r > 3$, and let $f : A \longrightarrow X$ be a mapping with $f(0) = 0$ satisfying (16), (17). Then there exists a unique cubic derivation $D : A \longrightarrow X$ satisfying*

$$\|f(a) - D(a)\| \leq \frac{\delta}{2(2^r - 8)}\|a\|^r \tag{23}$$

for all $a \in A$.

Proof The result follows from Theorem 2 by putting

$$\varphi(a, b, c, d) = \delta(\|a\|^r + \|b\|^r + \|c\|^r + \|d\|^r).$$

3 A fixed point approach

In this section, we prove the stability and the superstability for cubic derivations on Banach algebras by using a fixed point theorem. First, we bring the following fixed point theorem which is proved in [6]. This theorem plays a fundamental role to achieve our purpose in this section (an extension of the result was given in [18]).

Theorem 3 (*The fixed point alternative*) *Let (Δ, d) be a complete generalized metric space and $\mathcal{J} : \Delta \longrightarrow \Delta$ be a mapping with a Lipschitz constant $L < 1$. Then, for each element $\alpha \in \Delta$, either $d(\mathcal{J}^n\alpha, \mathcal{J}^{n+1}\alpha) = \infty$ for all $n \geq 0$, or there exists a natural number n_0 such that:*

(i) *$d(\mathcal{J}^n\alpha, \mathcal{J}^{n+1}\alpha) < \infty$ for all $n \geq n_0$;*
(ii) *the sequence $\{\mathcal{J}^n\alpha\}$ is convergent to a fixed point β^* of $\mathcal{J}$;*
(iii) *β^* is the unique fixed point of $\mathcal{J}$ in the set $\Delta_1 = \{\beta \in \Delta : d(\mathcal{T}^{n_0}\alpha, \beta) < \infty\}$;*
(iv) *$d(\beta, \beta^*) \leq \frac{1}{1-L} d(\beta, \mathcal{J}\beta)$ for all $\beta \in \Delta_1$.*

Theorem 4 *Let $f : A \longrightarrow X$ be a continuous mapping with $f(0) = 0$ and let $\phi : A^2 \longrightarrow [0, \infty)$ be a continuous function such that*

$$\|f(2\lambda a + \lambda b) + f(2\lambda a - \lambda b) - 2\lambda^3 f(a + b) - 2\lambda^3 f(a - b) - 12\lambda^3 f(a)\|$$

$$\leq \phi(a, b) \tag{24}$$

$$\|f(ab) - f(a) \cdot b^3 - a^3 \cdot f(b)\| \leq \phi(a, b) \tag{25}$$

for all $\lambda \in \mathbf{T}_{\frac{1}{n_0}}$ *and all* $a, b \in A$*. If there exists a constant* $k \in (0,1)$*, such that*

$$\phi(2a, 2b) \leq 8k\phi(a, b) \tag{26}$$

for all $a, b \in A$*, then there exists a unique cubic derivation* $D : A \longrightarrow X$ *satisfying*

$$\|f(a) - D(a)\| \leq \frac{1}{16(1-k)}\phi(a, 0) \tag{27}$$

for all $a \in A$*.*

Proof To achieve our goal, we make the conditions of Theorem 3. We consider the set

$$\Delta = \{g : A \longrightarrow X |\ g(0) = 0\}$$

and define the mapping d on $\Delta \times \Delta$ as follows:

$$d(g, h) := inf\{c \in (0, \infty) : \|g(a) - h(a)\| \leq c\phi(a, 0), \quad (\forall a \in A)\},$$

if there exist such constant c, and $d(g, h) = \infty$, otherwise. Similar to the proof of [2, Theorem 2.2], we can show that d is a generalized metric on Δ and the metric space (Δ, d) is complete. Now, we define the mapping $\mathcal{J} : \Delta \longrightarrow \Delta$ by

$$\mathcal{J}h(a) = \frac{1}{8}h(2a), \quad (a \in A). \tag{28}$$

If $g, h \in \Delta$ such that $d(g, h) < c$, by definition of d and $\mathcal{J}$, we have

$$\left\|\frac{1}{8}g(2a) - \frac{1}{8}h(2a)\right\| \leq \frac{1}{8}c\phi(2a, 0)$$

for all $a \in A$. Applying (26), we get

$$\left\|\frac{1}{8}g(2a) - \frac{1}{8}h(2a)\right\| \leq ck\phi(a, 0)$$

for all $a \in A$. The above inequality shows that $d(\mathcal{J}g, \mathcal{J}h) \leq kd(g, h)$ for all $g, h \in \Delta$. Hence, $\mathcal{J}$ is a strictly contractive mapping on Δ with a Lipschitz constant k. Here, we prove that $d(\mathcal{J}f, f) < \infty$. Putting $b = 0$ and $\lambda = 1$ in (24), we obtain

$$\|2f(2a) - 16f(a)\| \leq \phi(a, 0)$$

for all $a \in A$. Hence

$$\|\frac{1}{8}f(2a) - f(a)\| \leq \frac{1}{16}\phi(a, 0) \tag{29}$$

for all $a \in A$. We conclude from (29) that $d(\mathcal{J}f, f) \leq \frac{1}{16}$. It follows from Theorem 3 that $d(\mathcal{J}^n g, \mathcal{J}^{n+1} g) < \infty$ for all $n \geq 0$, and thus in this Theorem we have $n_0 = 0$. Therefore the parts (iii) and (iv) of Theorem 3 hold on the

whole Δ. Hence there exists a unique mapping $D : A \longrightarrow X$ such that D is a fixed point of $\mathcal{J}$ and that $\mathcal{J}^n f \to D$ as $n \to \infty$. Thus

$$\lim_{n\to\infty} \frac{f(2^n a)}{8^n} = D(a) \tag{30}$$

for all $a \in A$, and so

$$d(f, D) \leq \frac{1}{1-k} d(\mathcal{J}f, f) \leq \frac{1}{16(1-k)}.$$

The above equalities show that (27) is true for all $a \in A$. Now, it follows from (26) that

$$\lim_{n\to\infty} \frac{\phi(2^n a, 2^n b)}{8^n} = 0. \tag{31}$$

Replacing a and b by $2^n a$ and $2^n b$ respectively in (24), we get

$$\|\frac{f(2^n(2\lambda a + b))}{8^n} - \frac{f(2^n(2\lambda a - b))}{8^n} - 2\lambda^3 \frac{f(2^n(a+b))}{8^n}$$

$$-2\lambda^3 \frac{f(2^n(a-b))}{8^n} - 12\lambda^3 \frac{f(2^n a)}{8^n}\| \leq \frac{\phi(2^n a, 2^n b)}{8^n}.$$

Taking the limit as $n \longrightarrow \infty$, we obtain

$$D(2\lambda a + \lambda b) + D(2\lambda a - \lambda b) = 2\lambda^3 D(a+b) + 2\lambda^3 D(a-b) + 12\lambda^3 D(a) \tag{32}$$

for all $a, b \in A$ and all $\lambda \in \mathbf{T}_{\frac{1}{\mathbf{n_0}}}$. By (32), D is a cubic mapping when $\lambda = 1$. Letting $b = 0$ in (32), we get $D(\lambda a) = \lambda^3 D(a)$ for all $a \in A$ and $\lambda \in \mathbf{T}_{\frac{1}{\mathbf{n_0}}}$. Similar to the proof of Theorem 1, we have $D(\lambda a) = \lambda^3 D(a)$ for all $a \in A$ and $\lambda \in \mathbf{T}$. Since D is a cubic mapping, $D(ra) = r^3 D(a)$ for any rational number r. It follows from the continuity of f and ϕ that for each $\lambda \in \mathbf{R}$, $D(\lambda a) = \lambda^3 D(a)$. The proof of Theorem 1 indicates that $D(\lambda a) = \lambda^3 D(a)$, for all $a \in A$ and $\lambda \in \mathbf{C}$ ($\lambda \neq \mathbf{0}$). Therefore, D is a cubic homogeneous map. If we replace a, b by $2^n a, 2^n b$ respectively in (25), we have

$$\|\frac{f(2^{2n} ab)}{8^{2n}} - \frac{f(2^n a)}{8^n} \cdot b^3 - a^3 \cdot \frac{f(2^n b)}{8^n}\| \leq \frac{\phi(2^n a, 2^n b)}{8^{2n}} \leq \frac{\phi(2^n a, 2^n b)}{8^n}.$$

for all $a, b \in A$. Taking the limit as n tend to infinity, we get $D(ab) = D(a) \cdot b^3 + a^3 \cdot D(b)$, for all $a, b \in A$. Therefore D is a unique cubic derivation.

In the following result, we get again Corollary 1 which is a direct consequence of the above Theorem.

Corollary 3 *Let p, δ be the nonnegative real numbers with $p < 3$ and let $f : A \Longrightarrow X$ be a mapping with $f(0) = 0$ such that*

$$\|f(2\lambda a + \lambda b) + f(2\lambda a - \lambda b) - 2\lambda^3 f(a+b) - 2\lambda^3 f(a-b) - 12\lambda^3 f(a)\|$$

$$\leq \delta(\|a\|^p + \|b\|^p) \tag{33}$$

$$\|f(ab) - f(a) \cdot b^3 - a^3 \cdot f(b)\| \leq \delta(\|a\|^p + \|b\|^p) \tag{34}$$

for all $\lambda \in \mathbf{T}_{\frac{1}{n_0}}$ and all $a, b \in A$. Then there exists a unique cubic derivation $D : A \longrightarrow X$ satisfying

$$\|f(a) - D(a)\| \leq \frac{\delta}{2(8 - 2^p)} \|a\|^p$$

for all $a \in A$.

Proof If we put $\phi(a, b) = \delta(\|a\|^p + \|b\|^p)$ in Theorem 4, we obtain the desired result.

In the next result, we show that under which conditions cubic derivations are superstable.

Corollary 4 *Let p, q, δ be non-negative real numbers with $0 < p + q < 3$ and let $f : A \longrightarrow X$ be a mapping with $f(0) = 0$ such that*

$$\|f(2\lambda a + \lambda b) + f(2\lambda a - \lambda b) - 2\lambda^3 f(a+b) - 2\lambda^3 f(a-b) - 12\lambda^3 f(a)\|$$

$$\leq \delta(\|a\|^p \|b\|^q) \tag{35}$$

$$\|f(ab) - f(a) \cdot b^3 - a^3 \cdot f(b)\| \leq \delta(\|a\|^p \|b\|^q) \tag{36}$$

for all $\lambda \in \mathbf{T}_{\frac{1}{n_0}}$ and all $a, b \in A$. Then f is a cubic derivation on A.

Proof Putting $a = b = 0$ in (35), we get $f(0) = 0$. Now, if we put $b = 0$, $\lambda = 1$ in (35), then we have $f(2a) = 8f(a)$ for all $a \in A$. It is easy to see by induction that $f(2^n a) = 8^n f(a)$, and so $f(a) = \frac{f(2^n a)}{8^n}$ for all $a \in A$ and $n \in \mathbf{N}$. It follows from Theorem 4 that f is a cubic homogeneous mapping. Letting $\phi(a, b) = \delta(\|a\|^p \|b\|^q)$ in Theorem 4, we can obtain the desired result.

Note that if a mapping $f : A \longrightarrow X$ satisfies the inequalities (35) and (36), where p, q and δ are non-negative real numbers such that $p + q > 3$ and p is greater than 0, then it is obvious that f is a cubic derivation on A by putting $\phi(a, b) = \delta\left(\|a\|^p \|b\|^q\right)$ in Theorem 4.

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References

1. A. Bodaghi, I. A. Alias and M. H. Ghahramani, Approximately cubic functional equations and cubic multipliers, J. Inequal. Appl., 2011, 53 (2011).
2. A. Bodaghi, I. A. Alias and M. H. Ghahramani, Ulam stability of a quartic functional equation, Abstr. Appl. Anal., Article ID 232630, 9 pages, doi:10.1155/2012/232630 (2012).
3. L. Cădariu and V. Radu, Fixed points and the stability of quadratic functional equations, An. Univ. Timişoara, Ser. Mat. Inform., 41, 25–48 (2003).
4. L. Cădariu and V. Radu, On the stability of the Cauchy functional equation: A fixed point approach, Grazer Math. Ber., 346, 43–52 (2004).
5. S. Czerwik, On the stability of the quadratic mapping in normed spaces, Abh. Math. Sem. Univ. Hamburg., 62, 59–64 (1992).
6. J. B. Diaz and B. Margolis, A fixed point theorem of the alternative for contractions on a generalized complete metric space, Bull. Amer. Math. Soc., 74, 305–309 (1968).
7. M. Eshaghi Gordji, A. Bodaghi, C. Park, A fixed point approach to the stability of double Jordan centralizers and Jordan multipliers on Banach algebras, U.P.B. Sci. Bull., Series A., 73, Iss. 2, 65–73 (2011).
8. M. Eshaghi Gordji, S. kaboli Gharetapeh, M. B. Savadkouhi, M. Aghaei and T. Karimi, On cubic derivations, Int. Journal of Math. Analysis, 4, No. 51, 2501–2514 (2010).
9. D. H. Hyers, On the stability of the linear functional equation, Proc. Nat. Acad. Sci. USA., 27, 222–224 (1941).
10. K. Y. Lee and J. R. Lee, Cubic double centralizers and cubic multipliers, Korean J. Math. 17, No. 4, 437–449 (2009).
11. A. Najati, Hyers-Ulam-Rassias stability of a cubic functional equation, Bull. Korean Math. Soc., 44, No. 4, 825–840 (2007).
12. S. Y. Jang and C. Park, Approximate $*$-derivations and approximate quadratic $*$-derivations on C^*-algebras, J. Inequal. Appl., 2011, 55 (2011).
13. C. Park, Fixed points and Hyers-Ulam-Rassias stability of Cauchy-Jensen functional equations in Banach algebras, Fixed Point Theory and Applications, 2007, Art. ID 50175 (2007).
14. C. G. Park and J. Hou, Homomorphisms between C^*-algebras associated with the Trif functional equation and linear derivations on C^*-algebras, J. Korean Math. Soc., 41, No. 3, 461–477 (2004).
15. J. M. Rassias, On approximately of approximately linear mappings by linear mappings, J. Funct. Anal., 46, 126–130 (1982).
16. Th. M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc., 72, 297–300 (1978).
17. K. Ravi, J. M. Rassias and P. Narasimman, Stability of a cubic fuctional equation in fuzzy normed space, J. Appl. Anal. Comput., 1, No. 3, 411–425 (2011).
18. M. Turinici, Sequentially iterative processes and applications to Volterra functional equations, Annales Univ. Mariae-Curie Sklodowska (Sect A)., 32, 127-134 (1978).
19. S. M. Ulam, Problems in Modern Mathematics, Chapter VI, Science Ed., Wiley, New York (1940).